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## LETTER TO THE EDITOR

# Field theoretic approaches to biconnectedness in percolating systems

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**Abstract.** Two field theoretic formulations for the percolation problem are presented from which the critical exponents describing the 'backbone' of the infinite cluster at the percolation threshold are obtained. At high spatial dimension,  $d$ , the order-parameter exponent for the backbone  $\beta^{(2)}$  is given by  $\beta^{(2)} = 2\beta + \psi^{(2)}\nu$ , where  $\beta$  is the critical exponent for the density of the infinite cluster and  $\psi^{(2)}$  is a new crossover exponent. In mean-field theory  $\psi^{(2)} = 0$  and for  $d = 6 - \epsilon$ ,  $\psi^{(2)} = 2\epsilon^2/49 + O(\epsilon^3)$ . Presumably,  $\psi^{(2)}(d)$  is a smooth function of  $d$  for  $d > d^*$ , where numerical and theoretical work indicates that  $d^*$  is about 3. Our result indicates that the fractal dimensionality of the backbone is given in terms of the percolation exponents as  $\gamma/\nu - \psi^{(2)}$ .

In a lattice in which bonds are randomly removed with probability  $1 - p$ , an infinite cluster of sites connected by occupied bonds exists for  $p$  greater than a critical value,  $p_c$ , and the system is said to percolate. (For reviews of percolation, see Stauffer (1979) and Essam (1980)). The situation for  $p > p_c$  may be described in terms of the degree of connectedness as follows. For  $p < 1$  there exists a fraction of sites which are in finite clusters. We call these sites '0-connected', in that they have 0 paths (over occupied bonds) to distant regions of the lattice. When all these 0-connected sites are removed, we are left only with sites in the infinite cluster, and these we will classify as ' $m$ -connected,' with  $m = 1, 2, 3, \dots$ . The probability  $P(p)$  that a site is part of the infinite cluster grows continuously from zero as  $(p - p_c)^\beta$ , where  $\beta$  is a critical exponent. If the occupied bonds are resistors and a potential difference is established between two ends of the sample for  $p > p_c$ , only a fraction of the bonds will conduct electricity. Those sites through which no current flows have the property that from them the maximum number of simultaneously independent paths to infinitely remote regions is 1. Here independent paths are those which have no bonds in common. These singly connected non-conducting sites are called '1-connected.' When these 1-connected sites are removed, the sites that remain comprise the 'backbone' of the infinite cluster (Kirkpatrick 1978). The backbone appears in the heuristic theories of percolation advanced by Skal and Shklovskii (1974) and de Gennes (1976) in which the infinite cluster is viewed as nodes connected by tortuous strands of occupied bonds to infinitely distant regions of the lattice. Hence the backbone is the 'biconnected' (Kirkpatrick 1978) part of the infinite cluster, and in our terminology these sites are  $m$ -connected, with  $m \geq 2$ . In fact, in the 'node-link' picture, the nodes are by

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implication those sites from which there emanate at least three independent paths to distant regions, and those we would call 3-connected. More recently, the 'node-link' picture has been elaborated by Coniglio (1982) to include consideration of the internal structure of nodes and strands which must occur at low dimensionality (Stanley 1977, Dasgupta *et al* 1978, Gefen *et al* 1981).

We now introduce probability densities,  $P^{(m)}$ , that a site is *at least*  $m$ -connected. Thus for  $p$  just larger than  $p_c$ ,

$$P^{(0)} \sim \text{constant} \quad P^{(1)} = P(p) \sim (p - p_c)^\beta. \quad (1a, b)$$

The probability  $P^{(2)}(p)$  that a site is a part of the backbone also increases continuously from zero for  $p > p_c$  according to (Kirkpatrick 1978)

$$P^{(2)}(p) \sim (p - p_c)^{\beta^{(2)}} \quad (1c)$$

and we shall also write for  $m > 1$

$$P^{(m)}(p) \sim (p - p_c)^{\beta^{(m)}}. \quad (1d)$$

Though there have been some numerical simulations (Kirkpatrick 1978, Li and Strieder 1982) of  $P^{(2)}(p)$  and also a real space renormalisation-group treatment (Shlifer *et al* 1979) of  $P^{(2)}(p)$ , there are as yet no concrete theoretical predictions regarding  $\beta^{(2)}$ . In particular, it has not been established whether  $\beta^{(2)}$  is a new totally independent percolation exponent, or whether it is simply related to the traditional percolation exponents. In this paper we will show that the higher-order connectedness exponents  $\beta^{(m)}$  for  $m > 1$  can be expressed in terms of new crossover exponents which are independent of the usual percolation exponents.

Associated with any order parameter  $M$ , one can define a two-point correlation function which tends to  $M^2$  at infinite spatial separation. The correlation function associated with  $P(p)$  is

$$G^{(1)}(x, x') = [\nu^{(1)}(x, x')]_p, \quad (2)$$

where  $[\ ]_p$  denotes an average over all configurations of occupied bonds and  $\nu^{(1)}(x, x')$  is unity if  $x$  and  $x'$  are connected by a path of occupied bonds and zero otherwise. Near  $p_c$ ,  $G^{(1)}(x, x')$  scales with  $x$  as

$$G^{(1)}(x, 0) = x^{-2\omega} f(x/\xi), \quad (3)$$

where  $\xi \sim |p - p_c|^{-\nu}$  and  $2\omega = d - 2 + \eta$ , where  $d$  is the spatial dimensionality. From this we conclude in the usual way that  $\beta = \omega\nu$  and that

$$\chi \equiv \sum_x G(x, 0) \sim |p - p_c|^{-\gamma}, \quad (4)$$

where  $\gamma = d\nu - 2\beta$ . Likewise, the correlation function associated with  $P^{(2)}(p)$  is

$$G^{(2)}(x, x') = [\nu^{(2)}(x, x')]_p \quad (5)$$

where  $\nu^{(2)}(x, x')$  is unity if the sites  $x$  and  $x'$  are connected by two independent paths of occupied bonds and zero otherwise. Near  $p_c$ , we expect  $G^{(2)}$  to obey a scaling law similar to (3):

$$G^{(2)}(x, x') = x^{-2\omega^{(2)}} f^{(2)}(x/\xi). \quad (6)$$

From this, we conclude that  $\beta^{(2)} = \omega^{(2)}\nu$  and that the susceptibility exponent satisfies  $\gamma^{(2)} = d\nu - 2\beta^{(2)}$ . More generally, one can define an  $m$ -connectedness function

$\nu^{(m)}(x, x')$  which is non-zero only if there are  $m$  independent paths connecting  $x$  and  $x'$ . The associated correlation function,  $G^{(m)}(x, x')$  scales like  $x^{-2\omega^{(m)}} f^{(m)}(x/\xi)$ , where  $\omega^{(m)} = \beta^{(m)}/\nu$ , where  $\beta^{(m)}$  is the exponent associated with the probability that a site is in an infinite  $m$ -connected cluster.

We will present here two different field theoretic methods for calculating  $G^{(2)}$ . One method is quite general, but algebraically complex. The other is algebraically simpler, but is not easily extended into the ordered phase. Both methods predict

$$G^{(m)}(x, x') = [G^{(1)}(x, x')]^m |x - x'|^{-2\psi^{(m)}}. \tag{7}$$

for  $p = p_c$ . Within an  $\epsilon$  expansion ( $\epsilon = 6 - d$ ) we find

$$\psi^{(m)} = m(m - 1)\epsilon^2/49 + O(\epsilon^3). \tag{8}$$

For  $m = 2$ , equation (7) implies that

$$\omega^{(2)} = 2\omega + \psi^{(2)} \quad \beta^{(2)} = 2\beta + \psi^{(2)}\nu \quad \gamma^{(2)} = 2\gamma - d\nu - 2\psi^{(2)}\nu. \tag{9a, b, c}$$

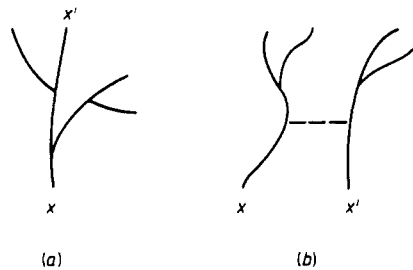
The algebraically simpler method of calculation is a modification of a field theory describing the statistics of branched polymers and gelation (Lubensky and Isaacson 1978). In this theory, there are fields  $\psi_j(x)$  carrying 'colours'  $j = 1, 2, \dots, s$ , sources  $w_i$ ,  $f$ -functional branching potentials, and repulsive potentials  $u$ . The Hamiltonian is

$$H = \int \left[ \sum_j \left( \frac{1}{2} r \psi_j^2 + \frac{1}{2} (\nabla \psi_j)^2 - \frac{1}{6} w_3 \psi_j^3 - w_1 \psi_j \right) + u \left( \sum_j \psi_j^2 \right)^2 \right] d^d x, \tag{10}$$

where for simplicity we consider only 3-functional branching. Each Feynman graph in the expansion of the partition function  $Z = \text{Tr} e^{-H}$  represents a configuration of polymer(s). The number of colours,  $s$ , is the polymer fugacity, and  $w_3$ ,  $w_1$  and  $r$  are related respectively to the fugacities for 3-functional units, free ends, and bifunctional monomers. Because of the source term proportional to  $w_1$ ,  $\langle \psi_j \rangle = Q$  is always non-zero and the propagator  $G_{ij}(x, x') \equiv \langle \psi_i(x) \psi_j(x') \rangle$  can be expressed as a sum of two terms

$$G_{ij}(x, x') = G_{\perp}(x, x') \delta_{ij} + R(x, x') = G_{\parallel}(x, x') + (\delta_{ij} - 1/s) G_{\perp}(x, x') \tag{11a, b}$$

where  $\delta_{ij}$  is the Kronecker delta. In (11a) the first term,  $G_{\perp}$ , represents propagation between endpoints on a single polymer, whereas the second,  $R$ , represents propagation between endpoints of different polymers. Representative diagrams for  $G_{\perp}$  and  $R$  are shown in figure 1. One sees that  $R$  is non-zero only if  $u$  is non-zero. In mean-field



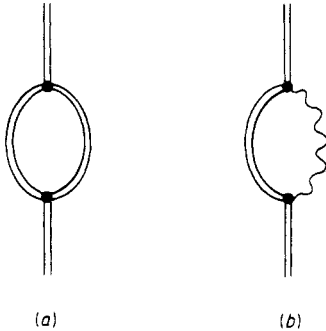
**Figure 1.** Mean-field (loopless) diagrams for the propagators  $G_{\perp}$  (a) and  $R$  (b). The lines represent propagation along a linear polymer segment (i.e.  $(r + q^2)^{-1}$ ), the broken line a repulsive interaction ( $u$ ), and the branch points the 3-functional fugacity,  $w_3$ .

theory we have

$$G_{\perp}(q)^{-1} = (r - wQ + 4usQ^2) + q^2 \quad G_{\parallel}^{-1}(q) = G_{\perp}^{-1}(q) + 8usQ^2 \quad (12a, b)$$

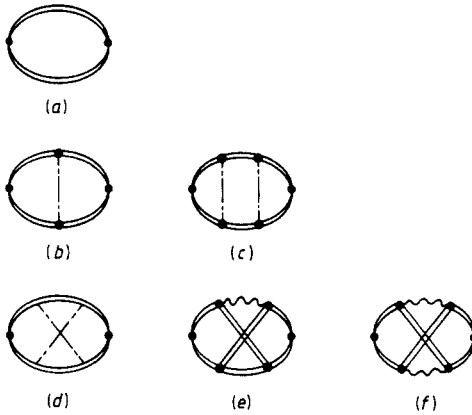
$$R(q) = s^{-1}(G_{\parallel}(q) - G_{\perp}(q)). \quad (12c)$$

At the gelation threshold,  $G_{\perp}(q=0)$  diverges, whereas  $G_{\parallel}(q=0)$  does not. Thus the dominant singularities associated with gelation can be obtained by replacing  $R(q)$  by  $-s^{-1}G_{\perp}(q)$ . On the other hand, in the dilute limit ( $s \rightarrow 0$ ),  $R(q) = -8G_{\perp}^2(q)uQ^2$ . In calculating  $G_{\perp}(q)$ , we must remain on a single polymer and there must be a continuous  $G_{\perp}$  line running from end to end. The two one-loop diagrams contributing to  $G_{\perp}(q)$  are shown in figure 2. The first represents a physical loop in the polymer whereas the second has a diagrammatic loop involving the potential  $u$  (via  $R$ ) which is not a physical loop of the polymer. Near percolation, figures 2(a) and (b) contribute  $(1 - 2s^{-1}) \int_q G_{\perp}^2(q)$  to  $G_{\perp}(0)$ , whereas in the dilute limit figure 2(b) dominates, contributing  $16uQ^2 \int_q G_{\perp}^3(q)$ . These rules can readily be used to calculate critical exponents for percolation (Harris *et al* 1975) and animals (Lubensky and Isaacson 1978) in an  $\epsilon$  expansion.



**Figure 2.** One loop diagrams contributing to  $G_{\perp}$ . The double line represents the mean-field  $G_{\perp}$  obtained by summing diagrams such as that of figure 1(a), whereas the wiggly line represents the mean-field  $R$ . Note that there is a continuous double line running from  $x$  to  $x'$  in both diagrams. (a) Represents a physical loop in the polymer, whereas (b) represents only a potential loop involving  $u$ . Near percolation, (a) carries a weight unity, whereas (b) carries a weight  $-2/s$ .

To calculate  $G^{(2)}(x, x')$ , we need two independent paths on the same polymer between  $x$  and  $x'$ . Representative diagrams contributing to  $G^{(2)}$  are shown in figure 3. Figure 3(a) leads to the mean-field result,  $G^{(2)}(x, x') = [G^1(x, x')]^2$ , which gives the dominant behaviour for  $d > 6$ . To order  $\epsilon$  (i.e. to order  $w_3^2$ ) we see that in the Feynman diagrams of figure 3(b) and (c), the cross-link which describes propagation between different independent paths always involves the combination  $G_{\perp} + R$ , which near gelation is  $(1 - s^{-1})G_{\perp}$  plus the non-singular term  $s^{-1}G_{\parallel}$ . In the limit  $s \rightarrow 1$ , corresponding to percolation with an unrestricted number of clusters, the singular term in  $G_{\perp} + R$  vanishes and we conclude that diagrams 3(b) and (c) do not contribute to the dominant behaviour for  $G^{(2)}$ . To order  $\epsilon^2$  we consider the diagrams having the topology of figure 3(d) involving either  $G_{\perp}$  or  $R$ . Simply to associate a factor  $G_{\perp} + R$  with each cross link would be incorrect, as it would overcount the diagram made with all  $G_{\perp}$ 's. A correct way to count the diagrams is to group them as in figures



**Figure 3.** Diagrams contributing to  $G^{(2)}(x, x')$ . Figure (a) is the lowest order contribution  $[G(x, x')]^2$ . The broken lines in (b)–(d) represent the combination  $G_{\perp} + R$ , the most singular part of which vanishes for  $s = 1$ . Thus these diagrams do not contribute to  $\gamma^{(2)}$ . (e) and (f) carry respective weights  $-2/s$  and  $1/s$  near threshold for a total weight from this topology of  $-1$  at  $s = 1$ . Since all these diagrams have three loops, they all carry an additional factor  $\frac{1}{4}$  due to symmetry.

3(d), (e), and (f). Thereby we obtain the result

$$G^{(2)}(x, x') = \frac{1}{2}G^{(1)}(x, x')^2 - \frac{1}{4}\Gamma(x, x'), \tag{13}$$

where  $\Gamma$  is the appropriate integral over eight propagators. A detailed calculation, like that given by Ma (1976) for the specific heat of the  $\phi^4$  model, shows that the most singular part of the Fourier transformed function  $\Gamma(q)$  is given by  $\Gamma(q = 0) \sim \frac{1}{2}w_3^4 K_6^3 r (\ln r)^2$  where  $K_d$  is the usual phase-space factor in  $d$  dimensions and  $w_3$  assumes its fixed point value,  $w_3^2 = 2\varepsilon / (7K_d)$ , where  $\varepsilon = 6 - d \ll 1$ . Then the Fourier transformed version of (13) is evaluated as

$$G^{(2)}(q = 0) \sim \frac{1}{2}K_6 r \ln r [1 - \frac{1}{2}(\varepsilon - 3\eta)\nu\gamma^{-1} \ln r + \frac{1}{4}w_3^4 K_6^2 \ln r] + A, \tag{14a}$$

where  $A$ , here, and  $B$  and  $C$ , below, are unimportant constants. By comparing this result with the expected form,

$$G^{(2)}(q = 0) \sim B + C \ln r [1 - \frac{1}{2}(1 + \gamma^{(2)}\gamma^{-1}) \ln r], \tag{14b}$$

we obtain (9c). Fourth and higher-order potentials are irrelevant. It is expected that these potentials remain irrelevant for  $d > d^*$ , where (Fucito and Parisi 1981)  $d^*$  is somewhat less than 3. A similar analysis shows that there will be singular corrections to figure 3(a) below  $d = 8$  for the dilute limit,  $s \rightarrow 0$ , which describes lattice animals (Lubensky and Isaacson 1979).

The second field theory for deriving (7) is based on the observation that for any thermal Hamiltonian,  $H$ , connecting spins  $s(x)$  (e.g. Ising spins) via exchanges over occupied bonds, the correlation function  $\langle s(x)s(x') \rangle$  is non-zero if and only if  $x$  and  $x'$  are in the same cluster. Thus for a given configuration of occupied bonds we have

$$\nu^{(1)}(x, x') = \lim_{k \rightarrow 0} \langle s(x)s(x') \rangle^k. \tag{15}$$

Taking the average over configurations we obtain

$$G^{(1)}(x, x') = \lim_{k \rightarrow 0} [\langle s(x)s(x') \rangle_p]^k = \lim_{k \rightarrow 0} \langle s^{\alpha_1}(x) \dots s^{\alpha_k}(x) s^{\alpha_1}(x') \dots s^{\alpha_k}(x') \rangle \quad (16a, b)$$

where  $\alpha$  is a replica index defined in the usual way, and the average in (16b) indicated by  $\langle \rangle$  is taken with respect to the averaged replicated Hamiltonian  $e^{-H} = [\prod_{\alpha} e^{-H_{\alpha}}]_p$ , and the limit  $n \rightarrow 0$  is taken. The Stephen and Grest (1977) treatment of the diluted Ising model can be extended to calculate  $G^{(1)}(x, x')$  as defined above and therefrom the properties of percolation.

To calculate  $G^{(2)}(x, x')$ , we need to add an additional index to distinguish different paths between  $x$  and  $x'$  and construct a Hamiltonian such that different paths do not have any occupied bonds in common. A suitable such Hamiltonian is defined by

$$e^{-H} = \prod_{NN} \left[ (1-p) + p \prod_{\alpha=1}^n \left( 1 + \lambda \sum_{\mu=1}^c \sigma_{\mu}^{\alpha}(x) \sigma_{\mu}^{\alpha}(x') \right) \right]. \quad (17)$$

Here  $\alpha$  is a replica index,  $\mu$  a new 'colour' index,  $\sigma_{\mu}^{\alpha}(x) = \pm 1$  is an Ising variable,  $NN$  indicates that the product is over pairs of nearest neighbouring sites,  $\lambda$  is a coupling constant, and  $c$  is the number of colours. When the partition function  $Z \equiv \text{Tr} e^{-H}$  is expanded in powers of  $p$  and  $(1-p)$ , each term may be identified with a particular configuration, since each unoccupied bond has weight  $(1-p)$  and each occupied bond a weight  $p$ . In addition, each occupied bond has a factor  $\lambda^m$  where  $m$  is the number of replicas appearing on that bond. Note that for each replica, a bond can have only one colour at a time. Thus it is straightforward to verify that

$$G^{(1)}(x, x') = \lim_{k \rightarrow 0} \left\langle \prod_{j=1}^k \sigma_{\mu_j}^j(x) \sigma_{\mu_j}^j(x') \right\rangle_H \quad (18a)$$

$$G^{(2)}(x, x') = \lim_{k \rightarrow 0} \left\langle \prod_{j=1}^k \sigma_{\mu_j}^j(x) \sigma_{\nu_j}^j(x) \sigma_{\mu_j}^j(x') \sigma_{\nu_j}^j(x') \right\rangle_H, \quad (18b)$$

where  $\mu_j$  and  $\nu_j$  must be different and  $\langle \rangle_H$  denotes an average with respect to  $e^{-H}$  given in (17). A check on this calculation is that when the  $k \rightarrow 0$  analytic continuations indicated in (18) are taken, the results no longer depend on  $\lambda$  or  $c$ , constants which have nothing to do with percolation.

A mean-field theory and an  $\epsilon$  expansion for these and higher connectedness functions can be developed from (17). Though algebraically somewhat tedious, this approach does allow a direct calculation of the order parameter

$$P^{(2)} = \lim_{k \rightarrow 0} \left\langle \prod_{j=1}^k \sigma_{\mu_j}^j(x) \sigma_{\nu_j}^j(x) \right\rangle_H$$

and we outline the method briefly. The field theory is written in terms of tensor fields  $Q_1$  with  $k$  upper and  $k$  lower indices which are conjugate to the operator  $\prod_{j=1}^k \sigma_{\mu_j}^j(x)$  and similar variables  $Q_2$  with  $2k$  upper and lower indices for the variables in (18b). For simplicity, we give here a schematic description of the theory, although detailed calculations require explicit consideration of the tensor indices. The Landau-Ginsburg-Wilson Hamiltonian is of the form

$$\int d^d x \left( \frac{1}{2} r_1 Q_1(x)^2 + \frac{1}{2} c_1 |\nabla Q_1(x)|^2 + \frac{1}{6} v_1 Q_1(x)^3 \right. \\ \left. + \frac{1}{2} r_2 Q_2(x)^2 + \frac{1}{2} c_2 |\nabla Q_2(x)|^2 + \frac{1}{2} v_2 Q_2(x) Q_1(x)^2 \right)$$

where  $r_1 \sim (p_c - p)$  and  $r_2 \sim 1$ . The momentum shell recursion relations (for  $c_n = 1$ ) for  $Q_1$  are the same as for the  $s \rightarrow 1$  field theory for the Potts model (Priest and Lubensky 1976):

$$r'_1 = b^{2-\eta_1}(r_1 - r_1 v_1^2 K_6 \ln b) \quad c'_1 = b^{-\eta_1}(1 - \frac{1}{6} v_1^2 K_6 \ln b) \quad (20a, b)$$

$$v'_1 = b^{(\varepsilon - 3\eta_1)/2}(v_1 - 2v_1^3 K_6 \ln b). \quad (20c)$$

(In writing these relations we have only kept the terms needed here.) As expected, these recursion relations do not involve  $r_2$ ,  $c_2$  or  $v_2$ , which are determined by

$$r'_2 = b^{2-\eta_2}(r_2 + r_1 v_2^2 K_6 \ln b) \quad c'_2 = b^{-\eta_2}(1 + \frac{1}{6} v_2^2 K_6 \ln b) \quad (20d, e)$$

$$v'_2 = b^{(\varepsilon - 2\eta_1 - \eta_2)/2} v_2 (1 - \frac{1}{2} K_6^2 v_1^4 \ln b + O(\varepsilon^3)) \quad (20f)$$

where  $\eta_n$  is the scale factor associated with  $Q_n$ . We look for a fixed point with  $c'_2 = 1$  and  $r'_2 \rightarrow \infty$ . The condition that  $v_2$  have a fixed point value yields

$$\eta_2 = \varepsilon - 2\eta_1 - 2\psi^{(2)}. \quad (21)$$

Setting  $v_1^2 K_6 = 2\varepsilon/7$  we obtain  $\psi^{(2)}$  as given in (8). This result implies that  $G^{(2)}(k)$  is of the form

$$G^{(2)}(k) \sim (r_2)^{-1} + k^{2-\eta_2}(r_2)^{-2} \dots \quad (22)$$

The singular part of  $G^{(2)}(k=0)$  is given by

$$G_{\text{sing}}^{(2)}(k=0) \sim \xi^{-2+\eta_2} \sim \xi^{[2\gamma/\nu - d - 2\psi^{(2)}]} \quad (23)$$

from which one easily recovers the results of equation (8).

We briefly discuss the applicability of the 'node-link' picture (Skal and Shklovskii 1974, de Gennes 1976). In this picture, as applied to the dilute resistor network, the network is viewed as 'nodes' (triconnected points) separated by distances of order  $\xi$ , the percolation correlation length. The nodes are supposed to be connected by tortuous strands whose length, or more properly, whose resistance,  $L$ , diverges at the percolation threshold as  $L \sim (p - p_c)^{-\zeta}$ . According to this picture the conductivity is given by

$$\mu = (d - 2)\nu + \zeta. \quad (24)$$

This exponent can be calculated from a field theory without recourse to any specific geometrical assumptions (Dasgupta *et al* 1978, Stephen 1978), and in fact Wallace and Young (1978) have shown that  $\zeta$  is unity to all order in perturbation theory, which is presumed to be valid within the  $\varepsilon = 6 - d$  expansion. Indeed, (24) seems to be valid with  $\zeta = 1$  for  $d$  greater than about 3. For  $d = 2$ ,  $\zeta$  is definitely greater than unity ( $\mu = 1.27 \pm 0.04$  according to Li and Strieder (1982)). The situation for  $d = 3$  is unclear. The numerical situation will be reviewed in detail elsewhere (Harris 1983).

Now we discuss the implication of this work for the node-link picture. If one really believes that the strands connecting nodes are structureless, then one might propose the relation

$$P^{(2)} \sim L'/\xi^d, \quad (25)$$

where  $L'$  is the path length between nodes. Since this quantity is not necessarily the same as the resistance between nodes (Dasgupta *et al* 1978), its divergence at  $p_c$  may be governed by a different critical exponent:

$$L' \sim (p - p_c)^{-\zeta'}, \quad (26)$$



from which there follows

$$\beta^{(2)} = d\nu - \zeta'. \tag{27}$$

Alternatively, one might attempt to calculate  $P^{(2)}$  via

$$P^{(2)} \sim L'P^{(3)}, \tag{28}$$

since  $P^{(3)}$  is the density of nodes (triconnected vertices). Equation (28) leads to the relation

$$\beta^{(2)} = \beta^{(3)} - \zeta'. \tag{29}$$

Although (26)–(29) do work in mean-field theory, they fail even to first order in  $\epsilon$  where  $\beta^{(p)} = p\beta$ . Comparing (27) and (29) we would have  $\beta^{(3)} = d\nu$ , a relation which fails in first order in  $\epsilon$ . If one assumes that  $\zeta'$  is the same as the conductivity crossover exponent,  $\zeta$ , then one can verify that neither (27) nor (29) is valid to first order in  $\epsilon$ . Recently Alexander and Orbach (1982) have proposed a relation between various fractal dimensionalities which yields

$$t - (d - 2)\nu = \zeta = (\beta + \gamma)/2, \tag{30}$$

in contradiction to the result  $\zeta = 1$  of Wallace and Young (1978). Identifying this value with  $\zeta'$  in (27) also leads to a relation for  $\beta^{(2)}$  which fails to first order in  $\epsilon$ . All these relations fail badly in low dimension. From this we conclude that the ‘node-link’ picture should not be taken too literally.

Now we discuss briefly the numerical situation. We summarise existing data on  $\beta^{(2)}$  in table 1. Since  $\beta^{(2)} = 2\beta$  for a Cayley tree (Gefen *et al* 1981, Harris 1983), (9) must be true within mean-field theory. We have listed this result for  $d = 6$ , since this is the upper critical dimension for percolation (Toulouse 1974, Harris *et al* 1975). In view of the smallness of  $\psi^{(m)}$ , we tabulate  $\beta^{(2)}/2\beta$ , which our results give as  $1 + \psi^{(2)}\nu/(2\beta) \sim 1 + (6 - d)^2/98$ . The numerical results are consistent with this result, but comparison for  $\epsilon > 2$  is not very meaningful. It is interesting to observe that the plausible guess (Kirkpatrick 1978),  $\beta^{(2)}/2\beta = 1$ , is well satisfied in three or more dimensions. This simple guess fails badly in two dimensions. As we have said, our derivation breaks down when the fourth-order potential becomes relevant, presumably at  $d^* \approx 3$  (Fucito and Parisi 1981). The fourth-order potential becoming relevant may also be responsible for the anomalous behaviour of the conductivity scaling law of (24).

**Table 1.** Numerical values of exponents.

$d$	$\beta^{(2)}$	$\beta$	$\beta^{(2)}/(2\beta)$
2	$0.38 \pm 0.02^a$	$0.14^d$	$1.35 \pm 0.1$
3	$0.9 \pm 0.1^b$	$0.41 \pm 0.02^e$	$1.1 \pm 0.2$
4	$1.1 \pm 0.1^b$	$0.52 \pm 0.03^f$	$1.05 \pm 0.15$
6	$2^c$	$1^g$	1

<sup>a</sup> Li and Strieder (1982).

<sup>b</sup> Kirkpatrick (1978).

<sup>c</sup> Gefen *et al* (1981) and Harris (1983).

<sup>d</sup> Nienhuis *et al* (1980) and Pearson (1980) give  $\beta = \frac{5}{36}$ .

<sup>e</sup> Sur *et al* (1977)

<sup>f</sup> Kirkpatrick (1976).

<sup>g</sup> Fisher and Essam (1961).

Finally, we note that since the fractal dimensionality,  $d_f^{(m)}$  of the  $m$ -connected backbone is given by (Kirkpatrick 1978)

$$d_f^{(m)} = d - \beta^{(m)}/\nu, \quad (31)$$

our results indicate that

$$d_f^{(2)} = \gamma/\nu - \psi^{(2)}. \quad (32)$$

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